

AT LEAST HALF OF ALL GRAPHS SATISFY $\chi \leq \frac{1}{4}\omega + \frac{3}{4}\Delta + 1$

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ABSTRACT. We prove that for any graph G at least one of G or \bar{G} satisfies $\chi \leq \frac{1}{4}\omega + \frac{3}{4}\Delta + 1$. In particular, self-complementary graphs satisfy this bound.

1. INTRODUCTION

In [5] Reed conjectured that

$$(1) \quad \chi \leq \left\lceil \frac{\omega + \Delta + 1}{2} \right\rceil.$$

In the same paper he proved that there exists $\epsilon > 0$ such that

$$\chi \leq \epsilon\omega + (1 - \epsilon)\Delta + 1,$$

holds for every graph. The ϵ used in the proof is quite small (less than 10^{-8}).

We prove the following.

Main Result. *Let G be a graph. Then at least one of G or \bar{G} satisfies*

$$\chi \leq \frac{1}{4}\omega + \frac{3}{4}\Delta + 1.$$

To prove this we combine a result from [4] on graphs containing a doubly critical edge with the following lemma.

Key Lemma. *Every graph satisfies $\chi \leq \frac{\iota + \omega + \Delta + n + 2}{4}$.*

Here ι is the maximum number of singleton color classes appearing in an optimal coloring of the graph (formally defined below).

2. STINGINESS

In [4] it was shown that a doubly critical edge is enough to imply an upper bound on the chromatic number that is slightly weaker than Reed's conjectured upper bound.

Lemma 2.1. *If G is a graph containing a doubly critical edge, then*

$$\chi(G) \leq \frac{1}{3}\omega(G) + \frac{2}{3}(\Delta(G) + 1).$$

The following two lemmas were proved in [1] using matching theory results.

Lemma 2.2. *If G is a graph with $\chi(G) > \left\lceil \frac{|G|}{2} \right\rceil$, then*

$$\chi(G) \leq \frac{\omega(G) + \Delta(G) + 1}{2}.$$

Lemma 2.3. *If G is a graph with $\alpha(G) \leq 2$, then*

$$\chi(G) \leq \left\lceil \frac{\omega(G) + \Delta(G) + 1}{2} \right\rceil.$$

Lemma 2.4. *Let G be a graph for which every optimal coloring has all color classes of order at most 2. Then*

$$\chi(G) \leq \left\lceil \frac{\omega(G) + \Delta(G) + 1}{2} \right\rceil.$$

Proof. If $\alpha(G) \leq 2$, the result follows by Lemma 2.3. Hence we may assume that we have an independent set $I \subseteq G$ with $|I| \geq 3$. Put $H = G \setminus I$. Since G has no optimal coloring containing a color class of order ≥ 3 , we have $\chi(H) = \chi(G)$. Then

$$\chi(H) = \chi(G) \geq \frac{|G|}{2} = \frac{|H| + 3}{2} > \left\lceil \frac{|H|}{2} \right\rceil.$$

Hence, by Lemma 2.2, we have

$$\chi(G) = \chi(H) \leq \frac{\omega(H) + \Delta(H) + 1}{2} \leq \frac{\omega(G) + \Delta(G) + 1}{2}.$$

The lemma follows. \square

Definition 1. The *stinginess* of a graph G (denoted $\iota(G)$) is the maximum number of singleton color classes appearing in an optimal coloring of G . An optimal coloring of G is called *stingy* just in case it has the maximum number of singleton color classes.

Lemma 2.5. *Let G be a graph and H an induced subgraph of G . If $\chi(G) = \chi(G \setminus H) + \chi(H)$, then $\iota(G) \geq \iota(G \setminus H) + \iota(H)$.*

Proof. Assume that $\chi(G) = \chi(G \setminus H) + \chi(H)$. Then patching together any optimal coloring of $G \setminus H$ with any optimal coloring of H yields an optimal coloring of G . The lemma follows. \square

Lemma 2.6. *Let G be a graph. Then $\chi(G) \leq \frac{\iota(G) + |G|}{2}$.*

Proof. Let $C = \{I_1, \dots, I_m, \{s_1\}, \dots, \{s_{\iota(G)}\}\}$ be a stingy coloring of G . Since $|I_j| \geq 2$ for $1 \leq j \leq m$, we have $\chi(G) \leq \iota(G) + \frac{|G| - \iota(G)}{2} = \frac{|G| + \iota(G)}{2}$. \square

3. RESPECTFULLY GREEDY PARTIAL COLORINGS

Definition 2. Let G be a graph. A partial coloring C of G is called *r-greedy* just in case every color class has order at least r .

Definition 3. Let G be a graph. A partial coloring of C of G is called *respectful* just in case $\chi(G \setminus \cup C) = \chi(G) - |C|$.

Lemma 3.1. *Let G be a graph and C a respectful 3-greedy partial coloring of G with $|G \setminus \cup C|$ minimal. Then*

$$\chi(G) \leq \frac{\omega(G) + \Delta(G) + 1}{2} + \frac{|C| + 1}{2}.$$

Proof. Put $H = G \setminus \cup C$. By the minimality of $|H|$, every optimal coloring of H has all color classes of order at most 2. Thus, by Lemma 2.4, we have

$$\chi(H) \leq \frac{\omega(H) + \Delta(H) + 1}{2} + \frac{1}{2}.$$

Using the minimality of $|H|$ again, we see that every vertex of H is adjacent to at least one vertex in each element of C . Hence $\Delta(H) \leq \Delta(G) - |C|$. Putting it all together, we have

$$\begin{aligned} \chi(G) &= \chi(H) + |C| \\ &\leq \frac{\omega(H) + \Delta(H) + 1}{2} + \frac{1}{2} + |C| \\ &\leq \frac{\omega(H) + \Delta(G) - |C| + 1}{2} + \frac{1}{2} + |C| \\ &\leq \frac{\omega(G) + \Delta(G) - |C| + 1}{2} + \frac{1}{2} + |C| \\ &= \frac{\omega(G) + \Delta(G) + 1}{2} + \frac{|C| + 1}{2}. \end{aligned}$$

□

Key Lemma. *Every graph satisfies $\chi \leq \frac{\iota + \omega + \Delta + n + 2}{4}$.*

Proof. Let C be a respectful 3-greedy partial coloring of a graph G with $|G \setminus \cup C|$ minimal. Since $\chi(G \setminus \cup C) = \chi(G) - |C|$ we have $\iota(G \setminus \cup C) \leq \iota(G)$ (by Lemma 2.5). Applying Lemma 2.6 yields

$$\begin{aligned} \chi(G) &= \chi(G \setminus \cup C) + |C| \\ &\leq \frac{\iota(G) + |G| - |\cup C|}{2} + |C| \\ &\leq \frac{\iota(G) + |G| - |C|}{2}. \end{aligned}$$

Adding this inequality with the inequality in Lemma 3.1 gives

$$2\chi(G) \leq \frac{\iota(G) + \omega(G) + \Delta(G) + |G| + 2}{2}.$$

The lemma follows. □

4. THE MAIN RESULTS

Theorem 4.1. *Let G be a graph. Then at least one of the following holds,*

- (1) $\chi(G) \leq \frac{1}{3}\omega(G) + \frac{2}{3}(\Delta(G) + 1),$
- (2) $\chi(G) \leq \frac{\omega(G) + |G| + \Delta(G) + 3}{4}.$

Proof. Assume that (1) does not hold. Then, by Lemma 2.1, we have $\iota(G) < 2$. Applying the Key Lemma gives

$$\chi(G) \leq \frac{1 + \omega(G) + \Delta(G) + |G| + 2}{4}.$$

The theorem follows. □

Corollary 4.2. *Let G be a graph satisfying $\Delta \geq \frac{n}{2}$. Then G also satisfies*

$$\chi \leq \frac{1}{4}\omega + \frac{3}{4}(\Delta + 1).$$

Proof. By Theorem 4.1, G satisfies

$$\begin{aligned} \chi &\leq \max\left\{\frac{1}{3}\omega + \frac{2}{3}(\Delta + 1), \frac{\omega + n + \Delta + 3}{4}\right\} \\ &\leq \max\left\{\frac{1}{4}\omega + \frac{3}{4}(\Delta + 1), \frac{\omega + n + \Delta + 3}{4}\right\} \\ &\leq \max\left\{\frac{1}{4}\omega + \frac{3}{4}(\Delta + 1), \frac{\omega + 3\Delta + 3}{4}\right\} \\ &= \frac{1}{4}\omega + \frac{3}{4}(\Delta + 1). \end{aligned}$$

□

We would like to find an upper bound on the chromatic number that must hold for a graph or its complement. The previous corollary is not quite good enough for this purpose since it doesn't handle $\frac{n-1}{2}$ -regular graphs. Instead, we use the following.

Corollary 4.3. *Let G be a graph satisfying $\Delta \geq \frac{n-1}{2}$. Then G also satisfies*

$$\chi \leq \frac{1}{4}\omega + \frac{3}{4}\Delta + 1.$$

Proof. By Theorem 4.1, G satisfies

$$\begin{aligned} \chi &\leq \max\left\{\frac{1}{3}\omega + \frac{2}{3}(\Delta + 1), \frac{\omega + n + \Delta + 3}{4}\right\} \\ &\leq \max\left\{\frac{1}{4}\omega + \frac{3}{4}(\Delta + 1), \frac{\omega + n + \Delta + 3}{4}\right\} \\ &\leq \max\left\{\frac{1}{4}\omega + \frac{3}{4}(\Delta + 1), \frac{\omega + 3\Delta + 4}{4}\right\} \\ &= \frac{1}{4}\omega + \frac{3}{4}\Delta + 1. \end{aligned}$$

□

Since every graph satisfies $\Delta + \bar{\Delta} \geq \Delta + n - 1 - \Delta = n - 1$, combining the pigeonhole principle with Corollary 4.3 proves the following.

Main Result. *Let G be a graph. Then at least one of G or \bar{G} satisfies*

$$\chi \leq \frac{1}{4}\omega + \frac{3}{4}\Delta + 1.$$

5. SOME RELATED RESULTS

In [3] the following was proven.

Lemma 5.1. *If G is a graph with $\iota(G) > \frac{\omega(G)}{2}$, then*

$$\chi(G) \leq \frac{\omega(G) + \Delta(G) + 1}{2}.$$

Theorem 5.2. *Let G be a graph. Then at least one of the following holds,*

- (1) $\chi(G) \leq \frac{\omega(G)+\Delta(G)+1}{2},$
- (2) $\chi(G) \leq \frac{3}{8}\omega(G) + \frac{|G|+\Delta(G)+2}{4}.$

Proof. Assume that (1) does not hold. Then, by Lemma 5.1, we have $\iota(G) \leq \frac{\omega(G)}{2}$. Applying the Key Lemma gives

$$\chi(G) \leq \frac{\frac{\omega(G)}{2} + \omega(G) + \Delta(G) + |G| + 2}{4}.$$

The theorem follows. □

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